

Uniform definition of sets using relations and complement of Presburger Arithmetic.

Arthur Milchior

Université Paris-Est Créteil,
LACL (EA 4219), UPEC,
F-94010 Créteil, France
Arthur.Milchior@u-pec.fr

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Abstract

In 1996, Michaux and Villemaire considered integer relations R which are not definable in Presburger Arithmetic. That is, not definable in first-order logic over integers with the addition function and the order relation ($\text{FO}[\mathbb{N}, +, <]$ -definable relations). They proved that, for each such R , there exists a $\text{FO}[\mathbb{N}, +, <, R]$ -formula $\nu_R(x)$ which defines a set of integers which is not ultimately periodic, i.e. not $\text{FO}[\mathbb{N}, +, <]$ -definable.

It is proven in this paper that the formula $\nu(x)$ can be chosen such that it does not depend on the interpretation of R . It is furthermore proven that $\nu(x)$ can be chosen such that it defines an expanding set. That is, an infinite set of integers such that the distance between two successive elements is not bounded.

1 Introduction

This paper deals with first order logic over non-negative integers with the addition function and the order relation. This logic is denoted $\text{FO}[\mathbb{N}, +, <]$ and is also called Presburger Arithmetic. A few properties of Presburger arithmetic are now recalled.

By [Pre27], the logic $\text{FO}[\mathbb{N}, +, <]$ admits the elimination of quantifiers. In particular it implies that $\text{FO}[\mathbb{N}, +, <]$ is a decidable theory. It is known that sets definable in Presburger Arithmetic coincide with semilinear sets [GS66], that is, finite union of linear sets. A linear set of dimension $d \in \mathbb{N}$ is a set of the form $\{(n_0^0, \dots, n_{d-1}^0) + \sum_{i=1}^c m^i(n_0^i, \dots, n_{d-1}^i) \mid m^1, \dots, m^c \in \mathbb{N}\}$, where $c \in \mathbb{N}$ and $n_j^i \in \mathbb{N}$ for $i \in [c]$ and $j \in [d-1]$. More precisely, the $\text{FO}[\mathbb{N}, +, <]$ -definable sets are exactly the finite union of disjoint linear sets [Ito69]. For the particular case of $d = 1$, a set R of integers is $\text{FO}[\mathbb{N}, +, <]$ -definable if and only if it is

ultimately periodic sets. That is, if there exists a threshold t and a period $p > 0$ such that, for all $n > t$, $n \in R$ if and only if $n + p \in R$.

A characterization of $\text{FO}[\mathbb{N}, +, <]$ -definable sets of dimension d is given in [Muc03, Theorem 2] in terms of sets of dimension $d - 1$ and in terms of local properties. Another characterization of $\text{FO}[\mathbb{N}, +, <]$ -definable sets is given in [MV96, Theorem 5.1]. This characterization states that for any $d \in \mathbb{N}$, if a set $R \subseteq \mathbb{N}^d$ is not $\text{FO}[\mathbb{N}, +, <]$ -definable, then there exists a set S of integers which is $\text{FO}[\mathbb{N}, +, <, R]$ -definable and which is not $\text{FO}[\mathbb{N}, +, <]$ -definable. Furthermore, it can be assumed that S is expanding, that is, that the difference between two successive elements of S is unbounded.

The result [MV96, Theorem 5.1] is particularly useful to reduce the complexity of proofs. Instead of assuming that a relation over integers is not $\text{FO}[\mathbb{N}, +, <]$ -definable, it is possible to assume that a set of integers is $\text{FO}[\mathbb{N}, +, <]$ -definable.

In particular, Theorem [MV96, Theorem 5.1] serves to prove Cobham's [Cob69] and Semenov's [Sem77] Theorem (Cobham's Theorem being the case $d = 1$ of Semenov's Theorem). Semenov's Theorem is: "Let k and l be multiplicatively independant (i.e. have no common power apart from 1). If $R \subseteq \mathbb{N}^d$ is definable in $\text{FO}[\mathbb{N}, +, <, V_k]$ and in $\text{FO}[\mathbb{N}, +, <, V_l]$ then R is $\text{FO}[\mathbb{N}, +, <]$ -definable." Here V_m is the function which maps every nonzero natural number to the greatest power of m dividing it.

The "central idea" of [MV96, Theorem 5.1], as stated in [MV96, page 272], is the following. Let $R \subseteq \mathbb{N}^d$ be a relation which is not $\text{FO}[\mathbb{N}, +, <]$ -definable. Furthermore, assume that all sets of integers which are $\text{FO}[\mathbb{N}, +, <, R]$ -definable are ultimately periodic. It can be shown that the negation of Muchnik's characterization [Muc03, Theorem 2] of $\text{FO}[\mathbb{N}, +, <]$ -definable sets does not hold (i.e. that Muchnik characterization holds). Since Muchnik's characterization holds, it implies that R is $\text{FO}[\mathbb{N}, +, <]$ -definable, which contradicts the hypothesis.

A careful analysis of the proof of [MV96, Theorem 5.1] shows that the use of proof by contradiction can be avoided. Removing the proof by contradiction would lead to a method which, given a relation $R \subseteq \mathbb{N}^d$, allows to construct a formula $\nu_R(x) \in \text{FO}[\mathbb{N}, +, <, R]$, defining a set of integers which is not $\text{FO}[\mathbb{N}, +, <]$ -definable and which is expanding.

In this paper, we prove that this formula $\nu_R(x)$ can be chosen independently of R . That is, there exists a $\text{FO}[\mathbb{N}, +, <, R]$ -formula $\nu_d(x)$, such that, if R is not $\text{FO}[\mathbb{N}, +, <]$ -definable, then $\nu_d(x)$ defines an expanding set of integers, hence a set of integers which is not $\text{FO}[\mathbb{N}, +, <]$ -definable.

Standard definitions are recalled in section 2. Results related to $\text{FO}[\mathbb{N}, +, <]$ are recalled in section 3. The main theorem is stated and proved in section 4.

2 Definitions

In this section, definitions are recalled. To avoid ambiguity, the “=” symbol is used for mathematical equality. The symbol “:=” is used when terms are defined. And the equality relation in formulas is denoted by “ \doteq ”. Let \mathbb{Z} denote the set of integers, let \mathbb{N} denote the set of non-negative integers and let $\mathbb{N}^{>0}$ denote the set of positive integers. For $a \in \mathbb{Z}$, let $|a|$ denote the absolute value of a , that is, a if $a \in \mathbb{N}$ and $-a$ otherwise. For S a finite set of positive integers, let $\text{lcm}(S)$ denote the least common multiple of the element of S , that is, the least integer n such that, for each $i \in S$, i divides n .

For $a \in \mathbb{N}$, let $[a]$ denote $\{0, \dots, a\}$. For $d \in \mathbb{N}^{>0}$, let S^d denote the set of d -tuples of elements of S . Let bold letters denote d -tuples of variables, such as $\mathbf{x} \in \mathbb{N}^d$, which is an abbreviation for (x_0, \dots, x_{d-1}) . For $i \in [d-1]$, the variable x_i is called the i -th component of \mathbf{x} . Let $\max(\mathbf{x})$ denote $\max\{x_i \mid i \in [d-1]\}$, let $\min(\mathbf{x})$ denote $\min\{x_i \mid i \in [d-1]\}$ and let $\|\mathbf{x}\|$ denote $\sum_{i=0}^{d-1} x_i$, it is said to be the norm of \mathbf{x} .

Functions and relations are applied component-wise on d -tuples. In particular $\mathbf{x} < \mathbf{y}$ means that $x_i < y_i$, for all $i \in [d-1]$. Let $|\mathbf{x}|$ (respectively, $\mathbf{x} + \mathbf{y}$) denote the d -tuple $(|x_0|, \dots, |x_{d-1}|)$ (respectively, $(x_0 + y_0, \dots, x_{d-1} + y_{d-1})$). Let $\mathbf{f}(n)$ denote $(f_0(n), \dots, f_{d-1}(n))$ for $n \in \mathbb{N}$.

Definition 2.1 (Ultimately (m) -periodic). A set $R \subseteq \mathbb{N}$, is *ultimately m -periodic* if there exists an integer $t \in \mathbb{N}$ such that for all $n \geq t$, $n \in R$ if and only if $n + m \in R$. A set is said to be ultimately periodic if it is ultimately m -periodic for some $m \in \mathbb{N}^{>0}$. The least such integer t is called the threshold of R . The least such integer m is called the minimal period of R .

Definition 2.2 (Expanding set). A set $R \subseteq \mathbb{N}$, is *expanding* if it is infinite and if the distance between two successive integers belonging to R is not bounded.

2.1 First-order logic

In this section, the definitions concerning the logical formalism of this paper are introduced.

Definition 2.3 (Vocabulary). A *vocabulary* is a set of the form

$$\mathcal{V} = \{(R_i/d_i)_{i < n}, (f_i/d'_i)_{i < p}, (c_i)_{i < q}\},$$

where n , p and q are either integers or ω (the cardinality of the set of integers).

For $i < n$, the R_i is a *relation* symbol and its arity is d_i . For $i < p$, the f_i is a *function* symbol and its arity is d'_i . For $i < q$, the c_i is a *constant* symbol.

In this paper the value of p is 1, apart in Lemma 4.7, and the only function is the addition. The value of n is 1 or 2 relation. The relations considered in this paper are the order relation $<$ and a relation R of dimension $d \in \mathbb{N}^{>0}$.

Definition 2.4 (Structure). Let \mathcal{V} be a vocabulary. A \mathcal{V} -structure \mathcal{S} over the universe \mathbb{N} is a tuple

$$(\mathbb{N}, (R_i^{\mathcal{S}})_{i < n}, (f_i^{\mathcal{S}})_{i < p}, (c_i^{\mathcal{S}})_{i < q})$$

where $R_i^{\mathcal{S}} \subseteq \mathbb{N}^{d_i}$ for $i < n$, where $f_i^{\mathcal{S}} : \mathbb{N}^{d'_i} \rightarrow \mathbb{N}$ for $i < p$, and where $c_i^{\mathcal{S}} \in \mathbb{N}$ for $i < q$.

For every constant symbol x , and $c \in \mathbb{N}$, let $\mathcal{S}[x/c]$ denote the structure such that $x^{\mathcal{S}[x/c]} = c$, and $\varsigma^{\mathcal{S}[x/c]} = \varsigma^{\mathcal{S}}$ for all other symbols $\varsigma \in \mathcal{V} \setminus \{x\}$.

In this paper, we consider the standard interpretation of $+$ and $<$ over \mathbb{N} . The first-order logic used in this paper is now defined.

Definition 2.5 ($\text{FO}[\mathbb{N}, \mathcal{V}]$). The set of \mathcal{V} -terms is defined by the grammar:

$$t(\mathcal{V}) ::= c_i \mid f_i(t_0, \dots, t_{d'_i-1})$$

where c_i is a constant of \mathcal{V} , f_i is a function of \mathcal{V} and the t_j 's are \mathcal{V} -terms.

The first-order logic over the vocabulary \mathcal{V} , denoted by $\text{FO}[\mathbb{N}, \mathcal{V}]$, is defined by the grammar:

$$\text{FO}[\mathbb{N}, \mathcal{V}] ::= \exists x.\psi \mid \forall x.\psi \mid \neg\phi_0 \mid \phi_0 \wedge \phi_1 \mid \phi_0 \vee \phi_1 \mid R_i(t_0, \dots, t_{d_i-1}) \mid t_0 \doteq t_1$$

where the t_i 's are \mathcal{V} -term, R_i is a symbol belonging to \mathcal{V} , the ϕ_i 's are $\text{FO}[\mathbb{N}, \mathcal{V}]$ -formulas and ψ is a $\text{FO}[\mathbb{N}, \mathcal{V}, x]$ -formula.

The atomic formula $<(x, y)$ is denoted $x < y$. Let $\phi_0 \implies \phi_1$ be an abbreviation for $(\neg\phi_0) \vee \phi_1$ and let $\phi_0 \iff \phi_1$ be an abbreviation for $(\phi_0 \implies \phi_1) \wedge (\phi_1 \implies \phi_0)$. The dimension and the curly brackets are omitted in logics' notations. For instance $\text{FO}[\mathbb{N}, \{+/2, </2\}]$ is abbreviated in $\text{FO}[\mathbb{N}, +, <]$. Let ϕ be a $\text{FO}[\mathbb{N}, \mathcal{V}, x_0, \dots, x_{d-1}]$ -formula. Then $\phi(x_0, \dots, x_{d-1})$ is said to be an $\text{FO}[\mathbb{N}, \mathcal{V}]$ -formula with dimension d . The x_i 's for $i \in [d-1]$, are called the free variables and do not belong to \mathcal{V} . Given some \mathcal{V} -structure \mathcal{S} , the semantic of a $\text{FO}[\mathbb{N}, \mathcal{V}]$ -formula is defined recursively as usual.

Definition 2.6 (Definability). Let \mathcal{V} be a vocabulary and \mathcal{S} be a \mathcal{V} -structure. Let $d \in \mathbb{N}$ and $\phi(x_0, \dots, x_{d-1})$ be a $\text{FO}[\mathbb{N}, \mathcal{V}]$ -formula of dimension d . The formula ϕ is said to *define* the d -ary set $\{\mathbf{n} \in \mathbb{N}^d \mid \mathcal{S}[\mathbf{x}/\mathbf{n}] \models \phi(\mathbf{x})\}$ in \mathcal{S} . A set $R \subseteq \mathbb{N}^d$ is said to be $\text{FO}[\mathbb{N}, \mathcal{V}]$ -definable in \mathcal{S} if there exists $\phi(x_0, \dots, x_{d-1}) \in \text{FO}[\mathbb{N}, \mathcal{V}]$ such that $R = \phi(x_0, \dots, x_{d-1})^{\mathcal{S}}$.

2.2 Some notations

Some notations are introduced in this section in order to simplify creation of formulas. A notation is now introduced which allows to simplify the logical definitions of functions.

Notation 2.7. Let $d, d' \in \mathbb{N}$, and let \mathcal{V} be a vocabulary. Let:

$$\phi(x_0, \dots, x_{d-1}; y_0, \dots, y_{d'-1})$$

denote that, for every d -tuple $\mathbf{n} \in \mathbb{N}^d$, there exists exactly one d' -tuple $\mathbf{n}' \in \mathbb{N}^{d'}$ such that $\mathcal{S}[\mathbf{x}/\mathbf{n}][\mathbf{y}/\mathbf{n}'] \models \phi(x_0, \dots, x_{d-1}, y_0, \dots, y_{d'-1})$. Then the d' -tuple \mathbf{n}' is denoted by $\phi(\mathbf{n})$. More precisely, for $\psi(\mathbf{y})$ a formula with d' free variable, $\psi(\phi(\mathbf{n}))$ is an abbreviation for $\exists \mathbf{n}'. \phi(\mathbf{n}, \mathbf{n}') \wedge \psi(\mathbf{n}')$.

The following notation states that some variables are interpreted by the minimal value such that a formula holds.

Notation 2.8. Let F be a finite ordered set. Let $(\phi_i)_{i \in F}$ be a set of $\text{FO}[\mathbb{N}, \mathcal{V}]$ -formulas. Let $i \in F$. A $\text{FO}[\mathbb{N}, \mathcal{V}]$ -formula $\min_i \{\phi_i\}$ is introduced which states that ϕ_i holds and i is minimal with this property. Let:

$$\min_i \{\phi_i\} := \phi_i \wedge \bigwedge_{j \mid j < i} \neg \phi_j.$$

Similarly, let $\mathbf{x} = (x_0, \dots, x_{d-1})$ be a tuple of variables and let $\phi(\mathbf{x})$ be a $\text{FO}[\mathbb{N}, \mathcal{V}]$ -formula. A $\text{FO}[\mathbb{N}, \mathcal{V}, <]$ -formula $\min_{\mathbf{x}} \{\phi(\mathbf{x})\}$ is introduced, which states that $\phi(\mathbf{x})$ holds and \mathbf{x} is lexicographically minimal with this property. Let:

$$\min_{\mathbf{x}} \{\phi(\mathbf{x})\} := \phi(\mathbf{x}) \wedge \forall \mathbf{y}. \left\{ \left[\bigvee_{j=0}^{d-1} \left(y_j < x_j \wedge \bigwedge_{k=0}^{j-1} y_k \doteq x_k \right) \right] \implies \neg \phi(\mathbf{y}) \right\}.$$

Finally, for ϕ_i a family of $\text{FO}[\mathbb{N}, \mathcal{V}]$ -formulas, let $\min_{i, \mathbf{x}} \{\phi_i(\mathbf{x})\}$ be a $\text{FO}[\mathbb{N}, \mathcal{V}, <]$ -formula which states that $\phi_i(\mathbf{x})$ holds and (i, \mathbf{x}) is lexicographically minimal with this property. Let:

$$\min_{i, \mathbf{x}} \{\phi_i(\mathbf{x})\} := \min_i \{\exists \mathbf{y}. \phi_i(\mathbf{y})\} \wedge \min_{\mathbf{x}} \{\phi_i(\mathbf{x})\}.$$

An example of formula using this notation is now given.

Example 2.9. Let R be a unary relation symbol. Let $\phi(x) := \min_x \{R(x) \wedge \neg R(x+1)\}$. This formula state that x is the last element of the least sequence of successive elements of R .

Notations for implications and equivalences are standard. A notation of the form “if then else” is also needed. It is now introduced.

Notation 2.10. Let F be a finite set. Let ψ be a $\text{FO}[\mathbb{N}, \mathcal{V}]$ -formula. For $i \in F$, let $\phi_i(\mathbf{x})$ and $\chi_i(\mathbf{x})$ be $\text{FO}[\mathbb{N}, \mathcal{V}]$ -formulas. Let

$$\left\langle \bigvee_{i \in F} \exists \mathbf{x}. \phi_i(\mathbf{x}) \mid \chi_i(\mathbf{x}) \mid \psi \right\rangle$$

be a $\text{FO}[\mathbb{N}, \mathcal{V}]$ -formula which states that if there exists $i \in F$ and $\mathbf{n} \in \mathbb{N}^d$ such that $\phi_i(\mathbf{n})$ holds, then $\chi_i(\mathbf{n})$, otherwise ψ . Formally, the formula is:

$$\left\{ \bigvee_{i \in F} \exists \mathbf{x}. \phi_i(\mathbf{x}) \wedge \chi_i(\mathbf{x}) \right\} \vee \left\{ \bigwedge_{i \in F} \forall \mathbf{x}. \neg \phi_i(\mathbf{x}) \wedge \psi \right\}.$$

An example of formula using this notation is now given.

Example 2.11. The formula

$$\left\langle \bigvee_{i=3}^5 x \dot{=} i \mid \exists z. z \times i \dot{=} y \mid \exists z. 2z + 1 \dot{=} y \right\rangle$$

states that if x is 3, 4 or 5, then y is a multiple of x , otherwise y is odd.

In this paper, the two preceding notations are used together, stating that, if there are some $i \in F$ and $\mathbf{x} \in \mathbb{N}^d$ such that $\phi_i(\mathbf{x})$ holds, the minimal pair is considered in $\chi_i(\mathbf{x})$, otherwise the formula ψ is considered.

3 Some results about FO $[\mathbb{N}, +, <]$ -definable sets

In this section, theorems concerning FO $[\mathbb{N}, +, <]$ -definable sets of dimension $d > 0$ are recalled. The theorems concerning any positive dimension are given in Section 3.1, and the theorem concerning the dimension 1 are given in Section 3.2.

3.1 Positive dimension

The theorem given in this section is a variant of the characterization of FO $[\mathbb{N}, +, <]$ -definable relations given in [Muc03, Theorem 1]. This presentation of the theorem is inspired of [MV96, Theorem 5.5]. This characterization of FO $[\mathbb{N}, +, <]$ -definable sets consists in two properties, a local property and a recursive property. The recursive property of [MV96, Theorem 5.5] uses the notion of section, which is now defined.

Definition 3.1 (Section). Let the dimension d be at least 2, $R \subseteq \mathbb{N}^d$, $i \in [d-1]$ and $c \in \mathbb{N}$. Then the *section of R in $x_i = c$* , denoted by $\text{sec}(R; x_i = c)$, is the set of $(d-1)$ -tuples obtained from R by fixing the i th component to c , that is:

$$\text{sec}(R; x_i = c) := \{(x_0, \dots, x_{d-2}) \in \mathbb{N}^{d-1} \mid (x_0, \dots, x_{i-1}, c, x_i, \dots, x_{d-2}) \in R\}.$$

The sections of the addition relation are now given as examples.

Example 3.2. Let $R = \{(x_0, x_1, x_2) \mid x_0 + x_1 = x_2\}$. Its sections are now studied. Let $c \in \mathbb{N}$. One has:

$$\begin{aligned} \text{sec}(R; x_0 = c) &= \text{sec}(R; x_1 = c) = \{(n, n + c) \mid n \in \mathbb{N}\}, \\ \text{sec}(R; x_2 = c) &= \{(n, c - n) \mid n \leq c\}, \end{aligned}$$

The local property of [MV96, Theorem 5.5] uses the notion of cube which is now introduced.

Notation 3.3. Let $d \in \mathbb{N}$, $R \subseteq \mathbb{N}^d$, $\mathbf{x} \in \mathbb{N}^d$ and $k \in \mathbb{N}$. The R -cube at \mathbf{x} of size k , denoted by $C_R(\mathbf{x}, k)$, is defined as:

$$C_R(\mathbf{x}, k) := \{\mathbf{y} \in [k]^d \mid \mathbf{x} + \mathbf{y} \in R\}$$

The following lemma considers equality of cubes.

Lemma 3.4. Let $d \in \mathbb{N}$, and R be a d -ary relation symbol. Let \mathcal{S} be a $\{\mathbb{N}, +, <, R\}$ -structure. There exists a $\text{FO}[\mathbb{N}, +, <, R]$ -formula $\beta_d(\mathbf{x}, \mathbf{y}, k)$ which states that the cubes $C_{R^S}(\mathbf{x}, k)$ and $C_{R^S}(\mathbf{y}, k)$ are equal.

Proof. The formula is:

$$\beta_d(\mathbf{x}, \mathbf{y}, k) := \forall \mathbf{z}. [\max(\mathbf{z}) \leq k] \implies [R(\mathbf{x} + \mathbf{z}) \iff R(\mathbf{y} + \mathbf{z})], \quad (1)$$

where $\max(\mathbf{z}) < k$ denotes $\bigwedge_{i=0}^{d-1} z_i < k$. \square

The local property of [MV96, Theorem 5.5] also uses the notion of *shifting* a cube. This notion is now introduced.

Definition 3.5. Let $d \in \mathbb{N}$, $R \subseteq \mathbb{N}^d$, $\mathbf{x} \in \mathbb{N}^d$ and $\mathbf{r} \in \mathbb{Z}^d$. Then it is said that the pair (\mathbf{x}, k) can be *shifted by \mathbf{r}* in R if $C_R(\mathbf{x}, k) = C_R(\mathbf{x} + \mathbf{r}, k)$.

It is now explained how to state in first order logic that the pair (\mathbf{x}, k) can be shifted by \mathbf{r} in R .

Lemma 3.6. Let $d \in \mathbb{N}$, and R be a d -ary relation symbol. Let \mathcal{S} be a $\{\mathbb{N}, +, <, R\}$ -structure. There exists a $\text{FO}[\mathbb{N}, +, <, R]$ -formula $\sigma_d(\mathbf{r}, k, \mathbf{x})$ which states that the pair (\mathbf{x}, k) can be shifted by \mathbf{r} in R^S .

Proof. The formula is:

$$\sigma_d(\mathbf{r}, k, \mathbf{x}) := \beta_d(\mathbf{x}, \mathbf{x} + \mathbf{r}, k). \quad (2)$$

where β_d is the formula of Lemma 3.6. Note that $\mathbf{r} \in \mathbb{Z}^d$. Formally, the variables takes values in \mathbb{N} in our formalism. It is trivial to simulate such variables taking values in \mathbb{Z} by using twice as many variables $\mathbf{r}^0, \mathbf{r}^1 \in \mathbb{N}^d$ and considering \mathbf{r} as $\mathbf{r}^0 - \mathbf{r}^1$. \square

The notion of pairs which admits a shift whose norm is bounded by some constant s is now introduced.

Definition 3.7. Let $d \in \mathbb{N}^{>0}$, $R \subseteq \mathbb{N}^d$, $\mathbf{x} \in \mathbb{N}^d$, and $s \in \mathbb{N}$. If there exists $\mathbf{r} \in \mathbb{Z}^d \setminus (0, \dots, 0)$ such that $\max(|r|) \leq s$ and such that the pair (\mathbf{x}, k) can be shifted by \mathbf{r} in R , then the pair (\mathbf{x}, k) is said to be s -shiftable in R^S .

It is now explained how to state in first order logic that the pair (\mathbf{x}, k) is s -shiftable in R .

Lemma 3.8. Let $d \in \mathbb{N}$, and R be a d -ary relation symbol. Let \mathcal{S} be a $\{\mathbb{N}, +, <, R\}$ -structure. There exists a $\text{FO}[\mathbb{N}, +, <, R]$ -formula $\varsigma_d(s, k, \mathbf{x})$ which states that the pair (\mathbf{x}, k) is s -shiftable in R^S .

Proof. The formula is:

$$\varsigma_d(s, k, \mathbf{x}) := \exists \mathbf{r} \in \mathbb{Z}^d. \max(|\mathbf{r}|) \leq s \wedge \bigvee_{i=0}^{d-1} r_i \neq 0 \wedge \sigma_d(\mathbf{r}, k, \mathbf{x}). \quad (3)$$

where σ_d is the formula of Lemma 3.6. \square

A variant of Muchnik's theorem is now recalled.

Theorem 3.9 ([MV96, Theorem 5.5]). *Let $d \in \mathbb{N}^{>0}$ and $R \subseteq \mathbb{N}^d$. The following properties are equivalent;*

1. *The set R is $\text{FO}[\mathbb{N}, +, <]$ -definable.*
2. (a) *If the dimension d is at least 2, then all sections of R are $\text{FO}[\mathbb{N}, +, <]$ -definable and*
 (b) *there exists $s \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$, there exists $t \in \mathbb{N}$ such that, for all $\mathbf{c} \in \mathbb{N}^d$ with $t \leq \min(\mathbf{c})$, the pair (\mathbf{x}, k) can be shifted by s in R .*

Property (2b) is now commented. Intuitively, the integer s represents the bound on the norm of the shift, the integer k represents the size of the cube, the integer t represents a threshold and the d -tuple \mathbf{c} represents the lowest corner of the cubes considered. Property (2b) states that there exists a distance s such that for all sizes k of cubes, there exists cubes of size k whose components are arbitrary great and this cube is s -shiftable

Let us say a word about the difference between [Muc03, Theorem 2], [MV96, Theorem 5.5] and Theorem 3.9. The version of [Muc03, Theorem 2] considers a notion of periodicity, while [MV96, Theorem 5.5] and Theorem 3.9 consider a notion of shift. Each of those notion can be restated using the other notion. The version of [Muc03, Theorem 2] consider relations over \mathbb{Z} while [MV96, Theorem 5.5] and Theorem 3.9 only considers relations over \mathbb{N} . The condition about \mathbf{c} in Property (2b) of Theorem 3.9 is “ $t \leq \min(\mathbf{c})$ ”, while it is “ $t \leq \|\mathbf{c}\|$ ” in [Muc03, Theorem 2] and it is “ $t \leq \max(\mathbf{c})$ ” in [MV96, Theorem 5.5]. The proof of [MV96, Theorem 5.5] still holds when $t \leq \max(\mathbf{c})$ is replaced by $t \leq \min(\mathbf{c})$ or by $t \leq \|\mathbf{c}\|$.

One of the main interest of Theorem 3.9 is given in the following Theorem.

Theorem 3.10 ([Muc03, Theorem 2]). *There exists a $\text{FO}[\mathbb{N}, +, <, R]$ -formula μ_d such that, for every $\{\mathbb{N}, +, <, R\}$ -structure \mathcal{S} , $\mathcal{S} \models \mu_d$ if and only if $R^{\mathcal{S}}$ is $\text{FO}[\mathbb{N}, +, <]$ -definable.*

A corollary of Theorem 3.9 is now given.

Corollary 3.11. *Let $d \in \mathbb{N}^{>0}$ and $R \subseteq \mathbb{N}^d$. If R is not $\text{FO}[\mathbb{N}, +, <]$ -definable, then one of the two following statements hold:*

- (a) *the dimension is at least 2 and a section of R is not $\text{FO}[\mathbb{N}, +, <]$ -definable*
 or

(b) for every $s \in \mathbb{N}$, there exists $k(R, s) \in \mathbb{N}$ such that for every $t \in \mathbb{N}$ there exists $\mathbf{c}(R, s, t) \in \mathbb{N}^d$ with $t \leq \min(\mathbf{c}(R, s, t))$ such that the pair $(\mathbf{c}(R, s, t), k(R, s))$ is not s -shiftable in R^S .

Let \mathcal{S} be a $\{\mathbb{N}, +, <, R\}$ -structure such that R^S is not $\text{FO}[\mathbb{N}, +, <]$ -definable. Let $s, t \in \mathbb{N}$. There may exist many values $k(R^S, s)$ and $\mathbf{c}(R^S, s, t)$ for which Property (b) holds. In this paper, it is always assumed that $k(R^S, s)$ and $\mathbf{c}(R^S, s, t)$ represent the lexicographically minimal such values.

Two examples of applications of this corollary are now given.

Example 3.12. Let $R_0 = \{(x_0^2, x_1) \mid x_0, x_1 \in \mathbb{N}\}$. In this case, Property (a) of Corollary 3.11 clearly holds, for the section $x_1 = 0$.

Example 3.13. Let $R_1 = \{(x_0, x_1) \in \mathbb{N}^2 \mid x_1 \equiv 1 \pmod{2}, x_0 \leq x_1^2\}$. This set is pictured in Figure 1. In this case, Property (a) does not hold and Property

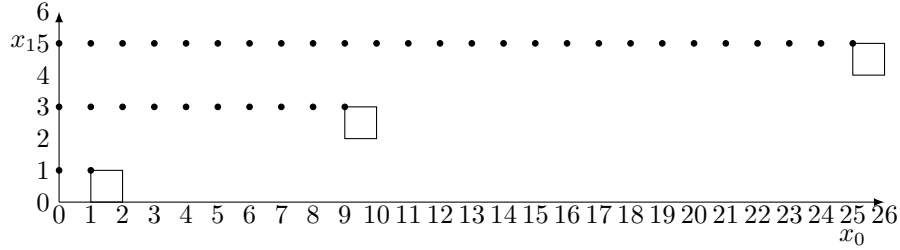


Figure 1: $R_1 = \{(x_0, x_1) \in \mathbb{N}^2 \mid x_1 \equiv 1 \pmod{2}, x_0 \leq x_1^2\}$ from Example 3.12

(b) holds. For every $s \in \mathbb{N}^{>0}$, it suffices to consider cubes of size 1, that is $k(R, s) = 1$. Indeed, there is an infinite number of $\mathbf{x} \in \mathbb{N}^d$ such that $C_R(\mathbf{x}, 1)$ equal to $\{(0, 1)\}$ and such that the pair $(\mathbf{x}, 1)$ is not s -shiftable in R^S . For small values of s , some of those cubes are shown in Figure 1.

More precisely, for every $t \in \mathbb{N}$ and for every $s \in \mathbb{N}^{>0}$, $\mathbf{c}(R^S, s, t)$ equals $((n+1)^2, n)$ where n is the least integer greater or equal to $\max(t, s/4)$.

The following lemmas allow to define the functions k and \mathbf{c} as first-order formulas which do not depend of the interpretation of R .

Lemma 3.14. Let $d > 0$, and R be a d -ary relation symbol. Let \mathcal{S} be a $\{\mathbb{N}, +, <, R\}$ -structure. There exists a $\text{FO}[\mathbb{N}, +, <, R]$ -formula $\kappa_d(s; K)$ which states that $K = k(R^S, s)$ if $k(R^S, s)$ is correctly defined.

Proof. The formula $\kappa_d(s; K)$ states that $K = k(R^S, s)$. The integer $k(R^S, s)$ is the minimal integer such that, for all $t \in \mathbb{N}$, there is a d -tuple $\mathbf{c} \in \mathbb{N}^d$ with $t \leq \min(\mathbf{c})$ such that the pair $(\mathbf{c}, k(R^S, s))$ is not s -shiftable in R^S . Let:

$$\kappa_d(s; K) := \min_K \{ \forall t. \exists \mathbf{c}. t \leq \min(\mathbf{c}) \wedge \neg \varsigma_d(s, K, \mathbf{c}) \}, \quad (4)$$

where $\varsigma_d(s, K, \mathbf{c})$ is the formula of Lemma 3.8 which states that the pair (\mathbf{c}, K) is s -shiftable in R^S . Recall that the notation $\min_K \{\phi\}$ is introduced in Notation 2.8. \square

Lemma 3.15. *Let $d > 0$, and R be a d -ary relation symbol. Let \mathcal{S} be a $\{\mathbb{N}, +, <, R\}$ -structure. There exists a $\text{FO}[\mathbb{N}, +, <, R]$ -formula $\gamma_d(s, t; \mathbf{C})$ which states that $\mathbf{C} = \mathbf{c}(R^{\mathcal{S}}, s, t)$ if $\mathbf{c}(R^{\mathcal{S}}, s, t)$ is defined.*

Proof. The formula $\gamma_d(s, t; \mathbf{C})$ states that \mathbf{C} is lexicographically minimal such that:

- $t \leq \min(\mathbf{C})$ and
- the pair $(\mathbf{C}, k(s))$ is not s -shiftable in $R^{\mathcal{S}}$.

Let

$$\gamma_d(s, t; \mathbf{C}) := \min_{\mathbf{C}} \left\{ \bigwedge_{i=0}^{d-1} t \leq C_i \wedge \neg \varsigma_d(s, \kappa_d(s), \mathbf{C}) \right\}, \quad (5)$$

where $\varsigma_d(s, \kappa_d(s), \mathbf{C})$ is the formula of Lemma 3.8. \square

3.2 Dimension $d = 1$

Two theorems dealing with set of integers and the logic $\text{FO}[\mathbb{N}, +, <]$ are recalled in this section.

Theorem 3.16 ([Pre27]). *A set $R \subseteq \mathbb{N}$ is $\text{FO}[\mathbb{N}, +, <]$ -definable if and only if it is ultimately periodic.*

Theorem 3.17 ([MV96, Theorem 3.7]). *Let R be a unary relation symbol. There exists a $\text{FO}[\mathbb{N}, +, <, R]$ -formula $\delta(x)$ such that, for all $\{\mathbb{N}, +, <, R\}$ -structure \mathcal{S} , if $R^{\mathcal{S}}$ is not ultimately periodic, then $\delta(x)^{\mathcal{S}}$ is expanding.*

Note that, formally, Theorem 3.17 is an easy consequence of [MV96, Theorem 3.7]. Indeed, [MV96, Theorem 3.7] states that there are two sets, which are $\text{FO}[\mathbb{N}, +, <, R]$ -definable, and one of them is expanding. And furthermore, the definition of those two sets does not depend on the interpretation of R .

4 The theorem

In this section, we prove the main theorem of this paper. It is similar to [MV96, Theorem 5.1], which is now recalled.

Theorem ([MV96, Theorem 5.1]). *Let $d \in \mathbb{N}^{>0}$ and $R \subseteq \mathbb{N}^d$. Then R is $\text{FO}[\mathbb{N}, +, <]$ -definable if and only if every subset of \mathbb{N} which is $\text{FO}[\mathbb{N}, +, <, R]$ -definable is ultimately periodic.*

This theorem can be equivalently stated as follows.

Corollary. *Let $d \in \mathbb{N}^{>0}$. Let \mathcal{S} be a $\{\mathbb{N}, +, <, R\}$ -structure such that $R^{\mathcal{S}} \subseteq \mathbb{N}^d$ which is not $\text{FO}[\mathbb{N}, +, <]$ -definable. There exists a $\text{FO}[\mathbb{N}, +, <, R]$ -formula $\nu_{R^{\mathcal{S}}}(x)$ such that $\nu_{R^{\mathcal{S}}}(x)^{\mathcal{S}}$ is not $\text{FO}[\mathbb{N}, +, <]$ -definable, i.e., not ultimately-periodic.*

Resuming Examples 3.12 and 3.13, two examples of such sets are given.

Example 4.1. Let $\mathcal{V} = \{\mathbb{N}, +, <, R\}$ where R is a binary relational symbol. Let \mathcal{S}^0 be the \mathcal{V} -structure such that

$$R^{\mathcal{S}^0} := \{(x_0^2, x_1) \mid x_0, x_1 \in \mathbb{N}\}.$$

In this case, it suffices to consider the section $x_1 = 0$. Then the formula $\nu(x) = R(x, 0)$, defines the set $\{n^2 \in \mathbb{N} \mid n \in \mathbb{N}\}$ which is not ultimately periodic.

Example 4.2. Let \mathcal{S}^1 be the \mathcal{V} -structure such that

$$R^{\mathcal{S}^1} := \{(x_0, x_1) \in \mathbb{N}^2 \mid x_1 \equiv 1 \pmod{2}, x_0 \leq x_1^2\}.$$

The set $R^{\mathcal{S}^1}$ is pictured in Figure 1. Each section of the form $x_0 = c$ is ultimately periodic with period 2 and each section of the form $x_1 = c$ is finite.

Let X be the set of pairs (x_0, x_1) such that $C_{R^{\mathcal{S}^1}}((x_0, x_1), 1) = \{(0, 1)\}$. The first of those elements are pictured as the lower-left corner of the squares of Figure 1. Then it can be shown that $X = \{((c+1)^2, c) \mid c \in 2\mathbb{N}\}$. Hence the set N of norms of elements of X is $\{c^2 + 3c + 1 \mid c \in 2\mathbb{N}\}$. Note that the set X is not ultimately periodic. The set N is defined by:

$$\begin{aligned} \nu(x) := \exists x_0, x_1. x_0 + x_1 &\doteq x \wedge \neg R(x_0, x_1) \wedge R(x_0, x_1 + 1) \\ &\wedge \neg R(x_0 + 1, x_1) \wedge \neg R(x_0 + 1, x_1 + 1). \end{aligned}$$

The main theorem of this paper is now stated.

Theorem 4.3. *Let $d \in \mathbb{N}^{>0}$. Let R be a d -ary relation symbol. There exists a FO $[\mathbb{N}, +, <, R]$ -formula $\nu_d(x)$ such that, for every $\{R, +, <\}$ -structure \mathcal{S} , if $R^{\mathcal{S}}$ is not FO $[\mathbb{N}, +, <]$ -definable, then $\nu_d(x)^{\mathcal{S}}$ is not ultimately-periodic, hence not FO $[\mathbb{N}, +, <]$ -definable.*

In order to prove this Theorem, two lemmas are first proven. The first lemma allows to reduce the problem of generating a set which is not ultimately periodic to a simpler case.

Lemma 4.4. *Let R be a binary relation and $\mathcal{V} = \{+, R\}$. There exists a FO $[\mathbb{N}, <, R]$ -formula $\epsilon(x)$ such that, for every \mathcal{V} -structure \mathcal{S} , if*

$$\begin{aligned} \text{for all } n \in \mathbb{N}, R_n := \{m \in \mathbb{N} \mid R^{\mathcal{S}}(n, m)\} &\text{ is ultimately periodic with} \\ \text{minimal period } p_n \in \mathbb{N}^{>0}, & \end{aligned} \quad (6)$$

and if :

$$\lim_{n \rightarrow +\infty} p_n = +\infty, \quad (7)$$

then $\epsilon(x)^{\mathcal{S}}$ defines a set $E(R^{\mathcal{S}})$ which is not ultimately periodic.

Two examples of sets R satisfying the hypothesis of this lemma are now given.

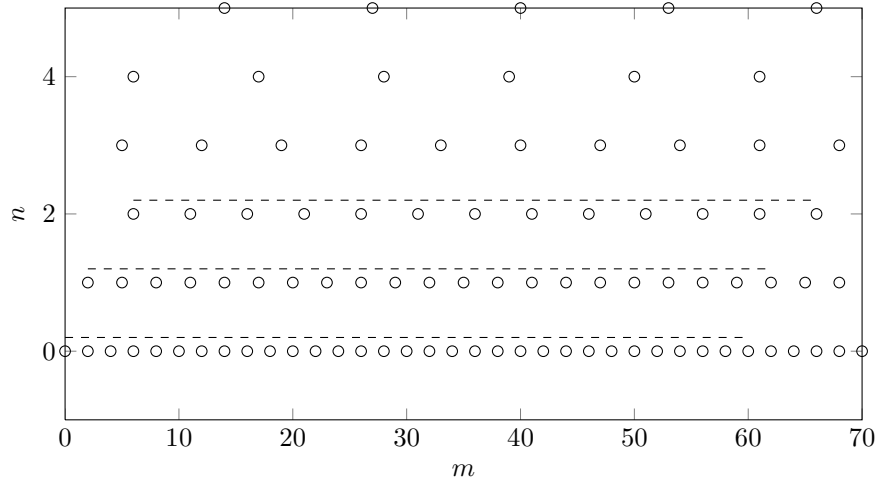


Figure 2: The set R of Example 4.6.

Example 4.5. Let π_n be the n -th prime integer. Let

$$R = \{(n, m) \in \mathbb{N}^2 \mid \pi_n \text{ divides } m\}.$$

Then $R_n = \{m \in \mathbb{N} \mid \pi_n \text{ divides } m\}$ is the set of multiple of π_n , its minimal periodicity p_n is π_n . Thus $\lim_{n \rightarrow +\infty} p_n = \lim_{n \rightarrow +\infty} \pi_n = \infty$. Let $q_n = \prod_{i=0}^n \pi_i$, it is the least positive integer such that $q \in R_i$ for all $i < n$. The distance between q_n and q_{n+1} is greater than π_{n+1} , thus is not bounded. Hence the set $S = \{q_n \mid n \in \mathbb{N}\}$ is not ultimately periodic. Note that the property $y = q_n$ is defined by the formula:

$$\rho(n; y) := \min_y \{0 < y \wedge \forall i \leq n. R(i, y)\}.$$

Recall that the notation $\min_y \{\phi\}$ is introduced in Notation (2.8). Finally, the set S is defined by:

$$\exists n. x \doteq \rho(n).$$

A second example is now given, which is a variation of the first example.

Example 4.6. Let $R = \{(n, m) \in \mathbb{N}^2 \mid \pi_n \text{ divides } m + n^2, m > n\}$. It is represented in Figure 2. Let $R_n = \{m \in \mathbb{N} \mid R(n, m)\}$. It is equal to $(m\mathbb{Z} - n^2) \cap \mathbb{N}$ and its minimal period p_n is also π_n . A formula $\alpha(n, p)$ is now introduced, which states that R_n is ultimately p -periodic. Let:

$$\alpha(n, p) := \exists t. \forall N > t. [R(n, N) \iff R(n, N + p)],$$

where t represents the threshold, as defined in Definition 2.2. Let $q_n = \prod_{i=0}^n \pi_i$, it is equal to $\text{lcm}\{\pi_i \mid i \in [n]\}$, hence to $\text{lcm}\{p_i \mid i \in [n]\}$. Thus q_n can be defined by:

$$\rho(n; q_n) := \min_{q_n} \{\forall i \leq n. \alpha(n, q_n)\},$$

For example, $\pi_0 = 2$, $\pi_1 = 3$ and $\pi_2 = 5$, hence $q_2 = 60$. The dashed lines of Figure 2, of length 60, illustrates the fact that the first three lines are ultimately 60-periodic. Finally, the set $S = \{q_n \mid n \in \mathbb{N}\}$ can still be represented as:

$$\exists n.x \doteq \rho(n).$$

Lemma 4.4 is now proven.

Proof of Lemma 4.4. The set $E(R^S)$ is the set of least common multiples of the p_i 's, for $i \in [0, n]$ for $n \in \mathbb{N}$. It must be shown that this set is not ultimately periodic and that it is $\text{FO}[\mathbb{N}, <, R]$ -definable. Let us first show that it is not ultimately periodic. For any integer n , let:

$$q_n := \text{lcm} \{p_i \mid i \in [n]\}. \quad (8)$$

Note that $E(R^S) = \{q_n \mid n \in \mathbb{N}\}$. It follows from Definition (8) of q_n that $p_n \leq q_n$, hence $\lim_{n \rightarrow +\infty} p_n \leq \lim_{n \rightarrow +\infty} q_n$. Since furthermore by Hypothesis (7) $\lim_{n \rightarrow +\infty} p_n = +\infty$, thus $\lim_{n \rightarrow +\infty} q_n = +\infty$, hence:

$$\text{There exists infinitely many integers } n \text{ such that } q_{n+1} \neq q_n. \quad (9)$$

It follows from Definition (8) of q_n that, for all $n \in \mathbb{N}$:

$$q_{n+1} = \text{lcm} \{p_i \mid i \in [n+1]\} = \text{lcm}(\text{lcm} \{p_i \mid i \in [n]\}, p_{n+1}) = \text{lcm}(q_n, p_{n+1}).$$

Since $q_{n+1} = \text{lcm}(q_n, p_{n+1})$, for all $n \in \mathbb{N}$, q_{n+1} is either q_n or is greater than $2q_n$. Furthermore, by statement (9), there exists infinitely many integers n such that $q_{n+1} \neq q_n$. It follows that there exists infinitely many integers n such that $2q_n \leq q_{n+1}$. It implies that the set $E(R^S) = \{q_n \mid n \in \mathbb{N}\}$ is infinite and the distance between two successive elements is not bounded. Hence $E(R^S)$ is not ultimately periodic.

It remains to logically define $E(R^S)$. A formula $\alpha(n, p)$ which states that p is a periodicity of R_n is first defined. Let:

$$\alpha(n, p) := \exists t. \forall N. t < N \implies [R(n, N) \iff R(n, N + p)],$$

where t represents the threshold, as defined in Definition 2.2. It should be noted that, for an arbitrary finite set F , the value of $\text{lcm}(F)$ does not seem to be $\text{FO}[\mathbb{N}, +, <, F]$ -definable. In this case, q_n is equivalently defined as the least integer p such that for all $i \leq n$, the set R_i is ultimately p -periodic. That is, q_n is defined by the $\text{FO}[\mathbb{N}, +, <, R]$ -formula:

$$\rho(n; q_n) := \min_{q_n} \{\forall i \leq n. \alpha(n, q_n)\}.$$

Recall that the notation $\min_x \{\phi\}$ is introduced in Notation (2.8). Finally, the formula $\epsilon(x)$ which defines $E(R^S)$ is:

$$\epsilon(x) := \exists n.x \doteq \rho(n).$$

□

The second lemma allows to transform a sequence of d functions, diverging to infinity, into a sequence of d *increasing* functions diverging to infinity, by restricting the domain of the d function to an infinite set of integers.

Lemma 4.7. *Let $d \in \mathbb{N}$, let f_0, \dots, f_{d-1} be unary function symbols and let $\mathcal{V} = \{<, f_0, \dots, f_{d-1}\}$. There exists a $\text{FO}[\mathbb{N}, \mathcal{V}]$ -formula $\tau_d(t)$ such that, for every \mathcal{V} -structure \mathcal{S} , if:*

$$\lim_{t \rightarrow +\infty} f_i^{\mathcal{S}}(t) = +\infty \text{ for all } i \in [d-1] \quad (10)$$

then the set $\tau_d(t)^{\mathcal{S}}$ defines an infinite set $T \subseteq \mathbb{N}$ such that \mathbf{f} is increasing of T .

Two functions f_0 and f_1 are now given as example. It is then explained how to define a $\text{FO}[\mathbb{N}, +, <, f_0, f_1]$ -formula which defines an infinite set over which both f_0 and f_1 are infinite.

Example 4.8. Let f_0 be the function which sends the integer n to $m \times n$, where $0 < m \leq 10$ and $n \equiv m \pmod{10}$. The first integers $f_0(n)$ are:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$f_0(n)$	0	1	4	9	16	25	36	49	64	81	100	11	24	39	56	75	96	119	144	171	200

Let us construct an infinite set T_0 over which f_0 is increasing. Clearly, T_0 can be taken to be $10\mathbb{N} + i$ for any $i \in [9]$. Those ten sets $10\mathbb{N} + i$ are $\text{FO}[\mathbb{N}, +]$ -definable. Note that the set $10\mathbb{N} + 1$ has the property that, for all $n \in 10\mathbb{N} + 1$ and $n' \in \mathbb{N}$ if $n < n'$, then $f(n) < f(n')$.

A second example is now given. Let \mathcal{P} be the set of prime numbers. Then let $f_1(n) = \sum_{i \in \mathcal{P} \cap [n]} i$. The first integers $f_1(n)$ are:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$f_1(n)$	0	0	2	5	5	10	10	17	17	17	17	28	28	41	41	41	41	58	58	77	77

When f_1 is the only considered function, the set T can be taken to be \mathbb{P} . Note that this set is not $\text{FO}[\mathbb{N}, +, <]$ -definable. However, it can be defined as the set $\{x \mid \forall y. x < y. f(x) \neq f(y)\}$.

Note that f_0 is not increasing on T_1 . Indeed, 7 and 11 belong to T_1 while:

$$f(7) = 7 \times 7 = 49 > 11 \times 1 = f(11).$$

In order to consider simultaneously the functions f_0 and f_1 , it suffices to replace the definition of T_1 by restricting the element to belong to T_0 . That is, let T be $\{x \in T_0 \mid \forall y. x < y. f(x) \neq f(y)\}$.

Lemma 4.7 is now proven.

Proof of Lemma 4.7. The proof is by induction on d . For $d = 0$, $T = \mathbb{N}$ can be chosen. Note that \mathbb{N} is $\text{FO}[\mathbb{N}, \mathcal{V}]$ -defined by the formula τ_0 equal to $\exists x. x \doteq x$.

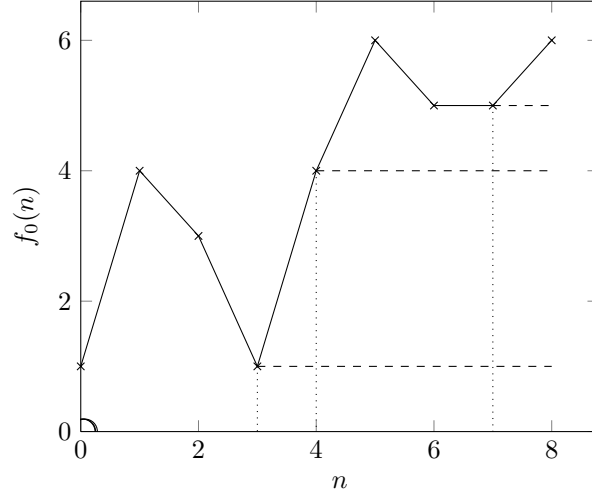


Figure 3: A function f and a set T such that T is increasing on f .

It is now assumed that $0 < d$ and that the property holds for $d - 1$. That is, it is assumed that there exists a $\text{FO}[\mathbb{N}, <, f_0, \dots, f_{d-2}]$ -formula $\tau_{d-1}(t)$ such that for every \mathcal{S} which satisfies the hypothesis, $\tau_{d-1}(t)^{\mathcal{S}}$ is a set $T' \subseteq \mathbb{N}$ such that:

$$T' \text{ is infinite} \quad (11)$$

and

$$\text{the functions } f_0, \dots, f_{d-2} \text{ are increasing on } T'. \quad (12)$$

Let T be the set of elements of T' such that the restriction of f_{d-2} on $T' \setminus [t - 1]$ is minimal on t . Geometrically speaking, T is the set of elements t such that the graph of f does not cross the lines on the right of $(t, f(t))$. It is illustrated in Figure 3 with $T' = \mathbb{N}$. The dashed horizontal lines are starting at $(t, f(t))$ for $t \in T$. The half-circles on the n axis represents the elements of T . Formally, let:

$$T := \{t \in T' \mid \forall t' \in T', t < t' \implies f_{d-1}(t) < f_{d-1}(t')\}. \quad (13)$$

Note that T is defined by the $\text{FO}[\mathbb{N}, \mathcal{V}]$ -formula:

$$\tau_d(t) := \tau_{d-1}(t) \wedge \forall t < t'. (\tau_{d-1}(t') \wedge t < t') \implies f_{d-1}(t) < f_{d-1}(t').$$

Let us prove that T satisfies the required condition, that is: \mathbf{f} is increasing on T and T is infinite. Let us first prove that \mathbf{f} is increasing on T . By Definition (13) of T , T is a subset of T' and by induction hypothesis (12), the functions f_0, \dots, f_{d-2} are increasing on T' . Thus:

$$\text{the functions } f_0, \dots, f_{d-2} \text{ are increasing on } T. \quad (14)$$

It remains to prove that f_{d-1} is increasing on T . Let:

$$t < t' \text{ be two elements of } T. \quad (15)$$

In order to prove that f_{d-1} is increasing on T , it remains to prove that $f_{d-1}(t)$ is smaller than $f_{d-1}(t')$. By Definition (15) of t' , $t' \in T$, and by Definition (13) of T , the elements of T belong to T' , thus:

$$t' \in T'. \quad (16)$$

By Definition (15) of t , $t \in T$ and $t < t'$, by Equation (16), $t' \in T'$ and by Definition (13) of T , for all $t' \in T'$ greater than t , $f_{d-1}(t) < f_{d-1}(t')$, thus $f_{d-1}(t) < f_{d-1}(t')$. Since $t < t' \in T$ implies that $f_{d-1}(t) < f_{d-1}(t')$, then:

$$f_{d-1} \text{ is increasing.} \quad (17)$$

Having both Statements (14) and (17) implies that:

$$f \text{ is increasing on } T. \quad (18)$$

It remains to prove that T is infinite. More precisely, it is proven that, for all $i \in \mathbb{N}$, T contains at least i elements. The proof is by induction on i . For $i = 0$, it is trivial. Let $i \in \mathbb{N}^{>0}$ and let us assume that:

$$\text{The set } T \text{ contains a subset } T_{i-1} \text{ of cardinality } i - 1. \quad (19)$$

In order to prove that T contains at least i elements, it suffices to define some integer t and prove that $t \in T \setminus T_{i-1}$. The integer t considered in the remaining of this proof is such that $\max T_{i-1} < t$, such that $f_{d-1}(t)$ is minimal under the preceding condition, and such that t is maximal under the preceding conditions. Using the example of Figure 3 with $i = 3$, $T_2 = \{3, 4\}$. The value of t is then 7. Geometrically, it is the one of lowest point $(t, f_{d-1}(t))$ belonging to the graph on f and on the right side of $(3, 1)$ and $(4, 4)$. Furthermore, between all of those minimal elements, it is the right-most one.

It must be proven that this minimum and this maximum exists and the integer satisfying this definition belongs to $T \setminus T_{i-1}$. Let us first prove that t is correctly defined. In the remaining of the proof, let $\max \emptyset = -1$. This assumption allows to avoid to consider the case $i = 1$ as special case. It is now proven that there exists a minimal integer c of the form $f_{d-1}(t)$ with $\max T_{i-1} < t$. By Hypothesis (10), $\lim_{t \rightarrow +\infty} f_{d-1}(t) = +\infty$ and by Induction hypothesis (11), T' is infinite, hence

$$\{t \mid t \in T', \max T_{i-1} < t, \max (f_{d-1}(T_{i-1})) < f_{d-1}(t)\}$$

is not empty, and thus the image of f_{d-1} on this set,

$$\{f_{d-1}(t) \mid t \in T', \max T_{i-1} < t, \max (f_{d-1}(T_{i-1})) < f_{d-1}(t)\},$$

is not empty. Since this set is a non-empty subset of \mathbb{N} , it contains a minimal element c . Formally, let:

$$c := \min \{f_{d-1}(t) \mid t \in T', \max T_{i-1} < t, \max(f_{d-1}(T_{i-1})) < f_{d-1}(t)\}. \quad (20)$$

As stated above, c is the minimal integer of the form $f_{d-1}(t)$ with $\max T_{i-1} < t$. It is now shown that there exists a maximal integer t , greater than $\max T_{i-1}$ such that $f_{d-1}(t) = c$. Note that it follows from Definition (20) of c that:

$$\max(f_{d-1}(T_{i-1})) < c. \quad (21)$$

By Definition (20) of c , it follows that:

$$\{t \mid t \in T', \max T_{i-1} < t, f_{d-1}(t) = c, \max(f_{d-1}(T_{i-1})) < f_{d-1}(t)\} \text{ is not empty.} \quad (22)$$

By Equation (21) $\max(f_{d-1}(T_{i-1})) < c$, thus having $f_{d-1}(t) = c$ implies that $\max(f_{d-1}(T_{i-1})) < f_{d-1}(t)$. Hence Statement (22) is equivalent to:

$$\{t \mid t \in T', \max T_{i-1} < t, f_{d-1}(t) = c\} \text{ is not empty.} \quad (23)$$

By Hypothesis (10), $\lim_{t \rightarrow +\infty} f_{d-1}(t) = +\infty$, thus there exists an integer $N \in \mathbb{N}$ such that for all $N < t$, $c < f_{d-1}(t)$. Hence N is an upper-bound of the set $\{t \mid t \in T', \max T_{i-1} < t, f_{d-1}(t) = c\}$. Since, furthermore, by Statement (23), this set is not empty, it admits a maximal element t . Formally, let:

$$t := \max \{t \mid t \in T', \max T_{i-1} < t, f_{d-1}(t) = c\}. \quad (24)$$

Note that:

$$c = f_{d-1}(t), \quad (25)$$

and that:

$$\max T_{i-1} < t. \quad (26)$$

Note that the integer t satisfies the properties stated in the beginning of the proof. It is greater than $\max T_{i-1}$, $f_{d-1}(t)$ is minimal under the preceding condition and t is maximal under the preceding conditions. It is now proven that $t \in T \setminus T_{i-1}$. By (26), $\max T_{i-1} < t$, hence:

$$t \notin T_{i-1}. \quad (27)$$

It remains to prove that $t \in T$. By Definition (13) of T , it suffices to prove that for all $t' \in T'$, $t < t'$ implies $f_{d-1}(t) < f_{d-1}(t')$. Let:

$$t' \in T' \text{ such that } t < t'. \quad (28)$$

By Equation (26) $\max T_{i-1} < t$ and by Statement (28) $t < t'$. Hence:

$$\max T_{i-1} < t'. \quad (29)$$

Since $T_{i-1} \subseteq T$:

$$\max T_{i-1} \in T. \quad (30)$$

By Statement (30), $\max T_{i-1} \in T$, by Equation (29), $\max T_{i-1} < t'$, thus by definition (13) of T :

$$f_{d-1}(\max(T_{i-1})) < f_{d-1}(t'). \quad (31)$$

By Statement (18), f_{d-1} is increasing on T and by Definition (19) of T_{i-1} , $T_{i-1} \subseteq T$, hence f_{d-1} is increasing on T_{i-1} . It follows that

$$f_{d-1}(\max(T_{i-1})) = \max(f_{d-1}(T_{i-1})). \quad (32)$$

Using this equality, $f_{d-1}(\max(T_{i-1}))$ can be replaced by $\max(f_{d-1}(T_{i-1}))$ in Equation (31). It follows that:

$$\max(f_{d-1}(T_{i-1})) < f_{d-1}(t'). \quad (33)$$

By Statement (24), t is the maximal element of T' , greater than $\max T_{i-1}$ and such that $f_{d-1}(t) = c$. Since, by definition (28) of t' , $t' \in T'$ and $t < t'$, and since, by Equation (29), $\max T_{i-1} < t$, it follows that:

$$c \neq f_{d-1}(t'). \quad (34)$$

By Statement (20), c is the minimal integer of the form $f_{d-1}(t)$, for $t \in T'$, with $\max T_{i-1} < t$, and $\max(f_{d-1}(T_{i-1})) < f_{d-1}(t)$. Since, by Definition (28) of t' , $t' \in T'$, since, by Equation (29), $\max T_{i-1} < t'$ and since, by Equation (33) $\max(f_{d-1}(T_{i-1})) < f_{d-1}(t')$, it follows that:

$$c \leq f_{d-1}(t'). \quad (35)$$

By Equation (34) $c \neq f_{d-1}(t')$ and by Equation (35) $c \leq f_{d-1}(t')$ then:

$$c < f_{d-1}(t'). \quad (36)$$

By Equation (36) $c < f_{d-1}(t')$ and by Equation (25), $c = f_{d-1}(t)$, thus: $f_{d-1}(t) < f_{d-1}(t')$. Since, for all $t' \in T'$ with $t < t'$, $f_{d-1}(t) < f_{d-1}(t')$, then:

$$t \in T. \quad (37)$$

Let $T_i = T_{i-1} \cup \{t\}$. By Equation (27), $t \notin T_{i-1}$, and by Definition (19) of T_{i-1} , T_{i-1} contains $i-1$ elements. Thus T_i contains i elements. By Definition (19) of T_{i-1} , the set T_{i-1} is a subset of T and by Equation (37), $t \in T$, thus $T_i \subseteq T$. Hence T admits a subset with i elements. Hence the induction hypothesis holds. \square

Theorem 4.3 is now proved.

Proof. The formula $\nu_d(x)$ is defined by induction on d . If $d = 1$, by Lemma 3.16, R^S is not ultimately periodic, hence it suffices to set $\nu_1(x) = R(x)$. Let us assume that d is at least 2 and that the theorem holds for $d-1$. Let:

$$\tilde{\mathcal{P}} \text{ be the set of structures } \mathcal{S} \text{ such that } R^S \text{ is not FO}[\mathbb{N}, +, <]\text{-definable.} \quad (38)$$

In this proof, many functions are defined. Their domain is $\tilde{\mathcal{P}} \times \mathbb{N}^{d'}$ for some $d' \in \mathbb{N}$ and their codomains are either the set of integers, the set of tuples of integers, or the set of subsets of \mathbb{N} . It is then shown that one of the set of integers is not ultimately periodic. When those functions are defined, $\text{FO}[\mathbb{N}, +, <, R]$ -formulas which define them are also given.

Let $\mathcal{S} \in \tilde{\mathcal{P}}$. By Definition (38) of $\tilde{\mathcal{P}}$, $R^{\mathcal{S}}$ is not $\text{FO}[\mathbb{N}, +, <]$ -definable. Thus, by Corollary 3.11 the structure \mathcal{S} satisfies one of the two properties of Corollary 3.11. Two cases must be considered, depending on which property of Corollary 3.11 is satisfied. Let us first assume that \mathcal{S} satisfies Property (a) of Corollary 3.11. That is, there exists $i \in [d-1]$ and $j \in \mathbb{N}$ such that $\text{sec}(R^{\mathcal{S}}; x_i = j)$ is not $\text{FO}[\mathbb{N}, +, <]$ -definable. In this case, the induction on d can be used on the lexicographically minimal pair (i, j) to generate the set $E(R^{\mathcal{S}})$. In Example 4.1, \mathcal{S}^0 satisfies Property (a) of Corollary 3.11. And in Example 4.1, \mathcal{S}^1 satisfies Property (b) of Corollary 3.11.

Let us define the formula ν_d . Let us assume that there exists a $\text{FO}[\mathbb{N}, +, <, R]$ -formula $\nu_{d,1}(x)$ such that, for all \mathcal{S} which satisfies Property (b) of Corollary 3.11, the set $\nu_{d,1}(x)^{\mathcal{S}}$ is not ultimately periodic. Let $\mu_{d,i}(s)$ be the $\text{FO}[\mathbb{N}, +, <, R]$ -formula which states that $\text{sec}(R^{\mathcal{S}}; x_i = s)$ is $\text{FO}[\mathbb{N}, +, <]$ -definable. It is defined by the formula μ_{d-1} of Theorem 3.10 where $R(n_0, \dots, n_{d-2})$ is replaced by $R(n_0, \dots, n_{i-1}, s, n_i, \dots, n_{d-2})$. Let $\nu'_{d-1,i}(x, s)$ be the formula $\nu_{d-1}(x)$ where $R(n_0, \dots, n_{d-2})$ is replaced by $R(n_0, \dots, n_{i-1}, s, n_i, \dots, n_{d-2})$ for all terms \mathbf{n} . Then let

$$\nu_d(x) := \left\langle \bigvee_{i=0}^{d-1} \exists s. \min_{i \in [d-1], s} \{ \neg \mu_{d,i}(s) \} \mid \nu'_{d-1,i}(x, s) \mid \nu_{d,1}(x) \right\rangle.$$

Recall that the notation $\min_x \{\phi\}$ is introduced in Notation (2.8) and that the notation $\langle \phi \mid \psi \mid \xi \rangle$ is introduced in Notation (2.10).

It remains to construct the $\text{FO}[\mathbb{N}, +, <, R]$ -formula $\nu_{d,1}(x)$. It is now assumed that the structure \mathcal{S} satisfies Property (b) of Corollary 3.11. That is:

For every $s \in \mathbb{N}$, there exists $k(R^{\mathcal{S}}, s) \in \mathbb{N}$ such that for every $t \in \mathbb{N}$, there exists $\mathbf{c}(R^{\mathcal{S}}, s, t) \in \mathbb{N}^d$ with $t \leq \min(\mathbf{c}(R^{\mathcal{S}}, s, t))$ such that the pair $(\mathbf{c}(R^{\mathcal{S}}, s, t), k(R^{\mathcal{S}}, s))$ is not s -shiftable in $R^{\mathcal{S}}$. (39)

As in Section 3, for s and t fixed, $k(R^{\mathcal{S}}, s)$ and $\mathbf{c}(R^{\mathcal{S}}, s, t)$ denote the lexicographically minimal tuple of integer which satisfies Statement (39). Recall that, by Lemmas 3.15 and 3.14, they are defined by the $\text{FO}[\mathbb{N}, +, <, R]$ -formulas $\gamma_d(s, t; \mathbf{c})$ and $\kappa_d(s; K)$ respectively. Note in particular that, by Statement (39), $t \leq \min(\mathbf{c}(R^{\mathcal{S}}, s, t))$, thus for all $s \in \mathbb{N}$ $\lim_{t \rightarrow +\infty} \min(\mathbf{c}(R^{\mathcal{S}}, s, t)) = +\infty$. Hence, for all $i \in [d-1]$ and for all $s \in \mathbb{N}$:

$$\lim_{t \rightarrow +\infty} (c_i(R^{\mathcal{S}}, s, t)) = +\infty. \quad (40)$$

By Statement (40), for each $s \in \mathbb{N}$, the d functions $c_0(R^{\mathcal{S}}, s, \cdot), \dots, c_{d-1}(R^{\mathcal{S}}, s, \cdot)$ satisfy the hypothesis of Lemma 4.7. Let:

$$T(R^{\mathcal{S}}, s) \subseteq \mathbb{N} \text{ be the set defined by Lemma 4.7 applied to the } d \text{ functions } c_0(R^{\mathcal{S}}, s, \cdot), \dots, c_{d-1}(R^{\mathcal{S}}, s, \cdot), \quad (41)$$

and let:

$$\begin{aligned} \tau_d(s, t) \text{ be the } \text{FO}[\mathbb{N}, +, <, R]\text{-formula of Lemma 4.7 where} \\ \text{each equality } y \doteq f_i(x) \text{ is replaced by the formula} \\ \exists z_0, \dots, z_{d-2}. \gamma_{d,i}(s, t; z_0, \dots, z_{i-1}, y, z_i, \dots, z_{d-1}). \end{aligned} \quad (42)$$

By Lemma 4.7, this formula defines $T(R^{\mathcal{S}}, s)$. The set $T(R^{\mathcal{S}}, s)$ is a set of indexes such that:

$$\text{for all } t, t' \in T(R^{\mathcal{S}}, s), t < t' \text{ implies } \mathbf{c}(R^{\mathcal{S}}, s, t) < \mathbf{c}(R^{\mathcal{S}}, s, t'). \quad (43)$$

The set $E(R^{\mathcal{S}})$ is extracted from $T(R^{\mathcal{S}}, s)$. For the structure \mathcal{S}^1 of Example 4.2, and for all $s \in \mathbb{N}^{>0}$, $T(R^{\mathcal{S}^1}, s) = 2\mathbb{N}$, and, as explained in Example 3.12, $\mathbf{c}(R^{\mathcal{S}^1}, s, t)$ is of the form $((c+1)^2, c)$.

The cubes of size s at position $\mathbf{c}(R^{\mathcal{S}}, s, t)$, are now considered. For $s, t \in \mathbb{N}$, let:

$$K(R^{\mathcal{S}}, s, t) \text{ denote the cube } C_{R^{\mathcal{S}}}(\mathbf{c}(R^{\mathcal{S}}, s, t), k(R^{\mathcal{S}}, s)). \quad (44)$$

The cube $K(R^{\mathcal{S}}, s, t)$ is such that all of its coordinates are at least t and furthermore the pair $(\mathbf{c}(R^{\mathcal{S}}, s, t), k(R^{\mathcal{S}}, s))$ is not s -shiftable in $R^{\mathcal{S}}$. For each $s \in \mathbb{N}$, by Definition (44), the set $\{K(R^{\mathcal{S}}, s, t) \mid t \in T(R^{\mathcal{S}}, s)\}$ is a set of subsets of $[k(s) - 1]^d$, hence is finite. It implies that for each $s \in \mathbb{N}$, there exists some cube $K(R^{\mathcal{S}}, s, t) \subseteq [k(s) - 1]^d$ that appears infinitely often in the sequence $(K(R^{\mathcal{S}}, s, t))_{t \in T(R^{\mathcal{S}}, s)}$. More precisely, it implies that:

$$\begin{aligned} \text{For each } s \in \mathbb{N}, \text{ there exists an integer } f(R^{\mathcal{S}}, s) \in T(R^{\mathcal{S}}, s) \text{ such} \\ \text{that } K(R^{\mathcal{S}}, s, f(R^{\mathcal{S}}, s)) \text{ appears infinitely often in the sequence} \\ (K(R^{\mathcal{S}}, s, t))_{t \in T(R^{\mathcal{S}}, s)}. \end{aligned} \quad (45)$$

Similarly to the choice of value of k and \mathbf{c} , the value of $f(R^{\mathcal{S}}, s)$ is chosen minimal. A $\text{FO}[\mathbb{N}, +, <, R]$ -formula $\phi(s; F)$ which states that $F = f(R^{\mathcal{S}}, s)$, as in Definition (45), is now given. By Lemma 3.6, there exists a $\text{FO}[\mathbb{N}, +, <, R]$ -formula $\beta_d(\mathbf{x}, \mathbf{y}, k)$ which states that the cube $C_{R^{\mathcal{S}}}(\mathbf{x}, k)$ is equal to the cube $C_{R^{\mathcal{S}}}(\mathbf{y}, k)$. Then, let:

$$\phi(s; F) := \min_F \{ \tau_d(s, F) \wedge \forall t \in \mathbb{N}. \exists t'. t < t' \wedge \tau_d(s, t') \wedge \beta_d(\mathbf{x}(s, F), \mathbf{x}(s, t'), \kappa_d(s)) \}.$$

A name is now given to this cube which appears infinitely often. Let:

$$I(R^{\mathcal{S}}, s) := K(R^{\mathcal{S}}, s, f(R^{\mathcal{S}}, s)). \quad (46)$$

For the structure \mathcal{S}^1 of Example 4.2, $f(R^{\mathcal{S}^1}, s) = 0$, for all $s \in \mathbb{N}^{>0}$, and

$$I(R^{\mathcal{S}^1}, s) = C_{R^{\mathcal{S}^1}}(C(R^{\mathcal{S}^1}, s, 0), 0) = \{(0, 1)\}.$$

The set $E(R^S)$ is extracted from the set $X(R^S, s)$ of indices of cubes equal to $I(R^S, s)$. Formally, let:

$$X(R^S, s) := \{t \in T(R^S, s) \mid K(R^S, s, t) = I(R^S, s)\}. \quad (47)$$

For the structure \mathcal{S}^1 of Example 4.1, for all $s \in \mathbb{N}^{>0}$, $X(R^{\mathcal{S}^1}, s) = T(R^{\mathcal{S}^1}, s) = 2\mathbb{N}$. A FO $[\mathbb{N}, +, <, R]$ -formula $\xi_d(s, t)$ is now given, which states that $t \in X(R^S, s)$. Let:

$$\xi_d(s, t) := \tau_d(s, t) \wedge \beta_d(\psi_d(s, t), \psi_d(s, \phi(s)), \kappa_d(s)).$$

By Definition (45) of $f(R^S, s)$, for all $s \in \mathbb{N}$:

$$\text{The set } X(R^S, s) \text{ is infinite.} \quad (48)$$

It follows from Statements (48) and (40) that, for all $s \in \mathbb{N}$:

$$\text{The set } \{c(R^S, s, t) \mid t \in X(R^S, s)\} \text{ is also infinite.} \quad (49)$$

The set $E(R^S)$ is constructed from $\{c(R^S, s, t) \mid t \in X(R^S, s)\}$. Note however that it is a set of tuples of integers and not a set of integers. In order to consider a set of integers, the set of norms is now considered. For all $s \in \mathbb{N}$, let

$$N(R^S, s) := \{\|c(R^S, s, t)\| \mid t \in X(R^S, s)\}. \quad (50)$$

For the structure \mathcal{S}^1 of Example 4.1, $N(s) = \{c + (c + 1)^2 \mid c \equiv 0 \pmod{2}, s/4 < c\}$. A FO $[\mathbb{N}, +, <, R]$ -formula $\zeta_d(s, x)$ is given, which states that $x \in N(R^S, s)$, as in Definition (50). Recall that $\gamma_d(s, t)$ is the formula of Lemma 3.15 which defines $c(R^S, s, t)$. Then let:

$$\zeta_d(s, x) := \exists t. \xi_d(s, t) \wedge x \doteq \|\gamma_d(s, t)\|, \quad (51)$$

Two cases must be considered, depending on whether there exists some s such that $N(R^S, s)$ is not ultimately periodic or whether for all s , $N(R^S, s)$ is ultimately periodic. If there exists an integer s such that $N(R^S, s)$ is not ultimately periodic, then, let $E(R^S)$ be $N(R^S, s)$. As usual, the integer s is assumed minimal.

The formula $\nu_{d,1}(x)$ is now defined. Let us assume that there exists a FO $[\mathbb{N}, +, <, R]$ -formula $\nu_{d,2}(x)$ such that $\nu_{d,2}(x)^S$ is not ultimately periodic, assuming that for all s , $N(R^S, s)$ is ultimately periodic. If there is $s \in \mathbb{N}$ such that some $N(R^S, s)$ is not ultimately periodic then $\nu_{d,1}(x)$ defines $N(R^S, s)$ with s minimal. Otherwise $\nu_{d,1}(x)$ uses the formula $\nu_{d,2}(x)$. Let $\mu'_1(s)$ be the formula which states that $N(R^S, s)$, it is the formula μ_1 of Theorem 3.10, where $R(x)$ is replaced by $\zeta_d(s, x)$. Finally, let:

$$\nu_{d,1}(x) := \left\langle \exists s. \min_s \{\neg \mu'_1(s)\} \mid \zeta_d(s, x) \mid \nu_d^2(x) \right\rangle.$$

It remains to construct the formula $\nu_{d,2}(x)$. It is now assumed that:

For all s , there exists an integer $p(R^S, s)$ such that $N(R^S, s)$ is ultimately $p(R^S, s)$ -periodic. (52)

Similarly to the choice of value of k , c and f , the integer $p(R^S, s)$ is the minimal integer which satisfies Statement (52). The set $E(R^S)$ is the set constructed by Lemma (4.4) applied to the set $\{(s, n) \mid n \in N(R^S, s)\}$. The FO $[\mathbb{N}, +, <, R]$ -formula $\nu_{d,2}(x)$ is just the formula $\epsilon(x)$ of Lemma 4.4, where $R(s, x)$ is replaced by the formula $\zeta_d(s, x)$.

It remains to prove that Lemma (4.4) can be applied to this set, that is, that $\lim_{s \rightarrow \infty} p(R^S, s) = \infty$. More precisely, it is proven that $s < p(R^S, s)$. In order to do this, it suffices to prove that $N(R^S, s)$ is infinite and that the distance between two distinct elements belonging to $N(R^S, s)$ is at least s . Let:

$$s \in \mathbb{N} \text{ and } t, t' \in X(R^S, s) \text{ such that } t < t'. \quad (53)$$

It follows, by Definition (47) of $X(R^S, s)$, that:

$$K(R^S, s, t) = K(R^S, s, f(R^S, s)) = K(R^S, s, t'). \quad (54)$$

By Definition (44) of $K(R^S, s, t)$, it implies:

$$C_{R^S}(c(R^S, s, t), c(R^S, s)) = C_{R^S}(c(R^S, s, t'), c(R^S, s)). \quad (55)$$

By Definition (39) of $c(R^S, s, t)$, the pair $(c(R^S, s, t), k(R^S, s))$ is not s -shiftable in R^S . Hence, by Equation (55):

$$c(R^S, s, t) = c(R^S, s, t') \text{ or } s < \max(|c(R^S, s, t') - c(R^S, s, t)|). \quad (56)$$

By Definition (53) of t and t' , $t, t' \in X(R^S, s)$, and by definition (47) of $X(R^S, s)$:

$$t, t' \in T(R^S, s) \quad (57)$$

By Statement (57), $t, t' \in T(R^S, s)$, by Definition (53) of t and t' , $t < t'$, by Statement (43), $t \mapsto c(R^S, s, t)$ is increasing on $T(R^S, s)$, thus:

$$c(R^S, s, t) < c(R^S, s, t'). \quad (58)$$

It follows trivially that:

$$c(R^S, s, t) \neq c(R^S, s, t'). \quad (59)$$

By Equation (59) and by Statement $c(R^S, s, t) \neq c(R^S, s, t')$ and by Statement (56) (either $c(R^S, s, t) = c(R^S, s, t')$ or $s < \max(|c(R^S, s, t') - c(R^S, s, t)|)$), hence:

$$s < \max(|c(R^S, s, t') - c(R^S, s, t)|). \quad (60)$$

By Equation (58) $c(R^S, s, t) < c(R^S, s, t')$, thus:

$$c(R^S, s, t') - c(R^S, s, t) = |c(R^S, s, t') - c(R^S, s, t)|. \quad (61)$$

By Equation (61) $(\mathbf{c}(R^S, s, t') - \mathbf{c}(R^S, s, t)) = |\mathbf{c}(R^S, s, t') - \mathbf{c}(R^S, s, t)|$, replacing $|\mathbf{c}(R^S, s, t') - \mathbf{c}(R^S, s, t)|$ by $\mathbf{c}(R^S, s, t') - \mathbf{c}(R^S, s, t)$ in Equation (60), it follows that $s < \max(\mathbf{c}(R^S, s, t') - \mathbf{c}(R^S, s, t))$, thus:

$$\text{There exists } i \in [d-1] \text{ such that } s < c_i(R^S, s, t') - c_i(R^S, s, t). \quad (62)$$

By Statement (58) $0 < c_j(R^S, s, t') - c_j(R^S, s, t)$ for all $j \in [d-1]$, and by Statement (62) $s < c_i(R^S, s, t') - c_i(R^S, s, t)$ for some $i \in [d-1]$. It follows that:

$$s < \sum_{i=0}^{d-1} c_i(R^S, s, t') - c_i(R^S, s, t). \quad (63)$$

Note that $\|\mathbf{c}(R^S, s, t')\| - \|\mathbf{c}(R^S, s, t)\| = \sum_{i=0}^{d-1} c_i(R^S, s, t') - c_i(R^S, s, t)$. Replacing $\sum_{i=0}^{d-1} c_i(R^S, s, t') - c_i(R^S, s, t)$ by $\|\mathbf{c}(R^S, s, t')\| - \|\mathbf{c}(R^S, s, t)\|$ in Equation (63) gives:

$$s < \|\mathbf{c}(R^S, s, t')\| - \|\mathbf{c}(R^S, s, t)\|. \quad (64)$$

By Statement (48), $X(R^S, s)$ is infinite, and by Statement (64), for t distinct from t' belonging to $X(R^S, s)$, $\|\mathbf{c}(R^S, s, t)\| \neq \|\mathbf{c}(R^S, s, t')\|$, thus:

$$\text{The set } N(R^S, s) \text{ is infinite.} \quad (65)$$

By Statement (65) $N(R^S, s)$ is infinite and by Statement (64), the difference between two integers of $N(R^S, s)$ is strictly greater than s . Hence:

$$N(R^S, s) \text{ is not ultimately } p\text{-periodic for any } p \leq s. \quad (66)$$

By Definition (52) of $p(R^S, s)$ and Statement (66), $s < p(R^S, s)$ for all $s \in \mathbb{N}$. Hence $\lim_{s \rightarrow \infty} p(R^S, s) = \infty$. It follows that the set $\{(s, i) \mid i \in N(R^S, s)\}$ satisfies the hypothesis of Lemma 4.4. It thus suffices to take $E(R^S)$ to be the set generated by this lemma and $\nu_{d,2}(x)$ to be the formula $\epsilon(x)$ of Lemma 4.4, where $R(s, x)$ is replaced by the formula $\zeta_d(s, x)$. \square

This theorem admits the following corollary.

Corollary 4.9. *Let $d \in \mathbb{N}^{>0}$. Let R be a d -ary relation symbol. There exists a $\text{FO}[\mathbb{N}, +, <, R]$ -formula $\nu'_d(x)$ such that, for every $\{R, +, <\}$ -structure \mathcal{S} , if R^S is not $\text{FO}[\mathbb{N}, +, <]$ -definable then $\nu'_d(x)^S$ is not expanding.*

Proof. It is straightforward from Proposition 4.3 and Theorem 3.17. \square

5 Conclusion

In this paper, we proved that any set R which is not $\text{FO}[\mathbb{N}, +, <]$ -definable allows to $\text{FO}[\mathbb{N}, +, <, R]$ -define an expanding set of integers, i.e. a set of integers which is not $\text{FO}[\mathbb{N}, +, <]$ -definable.

We see two directions for further research. The first direction consists in considering the same problem over other domains, such as the reals, the rationals, or the finite domains.

The second direction consists in considering the same problem for other vocabularies. In particular, the logic $\text{FO}[\mathbb{N}, \times, \text{is a divisor of}]$ on the domain \mathbb{N} is very similar to $\text{FO}[\mathbb{N}, +, <]$ on a domain of arbitrary dimension. Hence, it may be possible to $\text{FO}[\mathbb{N}, \times, \text{is a divisor of}]$ -define some interesting set of power of 2 using some sets which are not $\text{FO}[\mathbb{N}, \times, \text{is a divisor of}]$ -definable.

Let $\text{mod } m$ be the set of predicates of the form $x \equiv i \text{ mod } m$. By [Milss], for each set R which is not $\text{FO}[<, \text{mod } m]$ -definable, there exists a $\text{FO}[<, R]$ -formula which defines a set which is not $\text{FO}[<, \text{mod } m]$ -definable. It may be interesting to apply methods introduced in this paper to this logic, in order to obtain a formula independent from the interpretation of R .

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- μ_d - The formula which states that R is $\text{FO}[\mathbb{N}, +, <]$ -definable, 8
- $\nu_d(x)$ - The formula which defines a set which is not ultimately periodic, 11
- $R_i(t_0, \dots, t_{d_i-1})$, 4
- $\sigma_d(\mathbf{r}, k, \mathbf{x})$ - The pair (\mathbf{x}, k) can be shifted by \mathbf{r} in R^s , 7
- $\varsigma_d(s, k, \mathbf{x})$ - The pair (\mathbf{x}, k) can be shifted by s in R^S , 7
- $\text{sec}(R; x_i = c)$ - Section $x_i = c$ of R , 6
- Section of a subset of \mathbb{N}^d , 6
- Structure, 4
- $\tau_d(t)$ - the formula which defines a domain to d function such that those functions are increasing, 14
- Ultimately m -periodic, 3
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